Robust self-testing of quantum devices

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28 November 2017

Outline

- Bell nonlocality
- Self-testing
- Robust self-testing
- Summary and open questions

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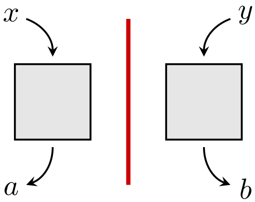
Bell nonlocality

Bell scenario



$$\Pr[a, b|x, y]$$

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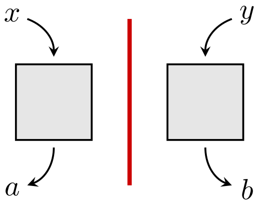

$\Pr[a, b|x, y]$

Def.: $\Pr[a, b|x, y]$ is **local** if

$$\Pr[a, b|x, y] = \sum_{\lambda} p(\lambda)\, p_A(a|x, \lambda)\, p_B(b|y, \lambda).$$

Otherwise $\implies$ **nonlocal** or it **violates (some) Bell inequality**

Bell nonlocality

Obs.: Separable states give local statistics (for all measurements)

$$\rho_{AB} = \sum_{\lambda} p_\lambda \sigma_\lambda \otimes \tau_\lambda,$$

$$\Pr[a,b|x,y] = \mathrm{tr}\left[(F_a^x \otimes G_b^y)\rho_{AB}\right] = \sum_{\lambda} p_\lambda \cdot \underbrace{\mathrm{tr}\left(F_a^x \sigma_\lambda\right)}_{p_A(a|x,\lambda)} \cdot \underbrace{\mathrm{tr}\left(G_b^y \tau_\lambda\right)}_{p_B(b|y,\lambda)}.$$

Bell nonlocality

$$\rho_{AB} \text{ is separable} \implies \text{statistics are } \textbf{local}$$

$$\Pr[a,b|x,y] \text{ is } \textbf{nonlocal} \implies \rho_{AB} \text{ is entangled}$$

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Question: can we make any more refined statements?

Bell nonlocality

ρ_{AB} is separable $\implies$ statistics are **local**

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Question: can we make any more refined statements?

Answer: self-testing

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Given $\Pr[a,b|x,y] = \mathrm{tr}\left[(F_a^x \otimes G_b^y)\rho_{AB}\right]$

deduce properties of ρ_{AB}, $\{F_a^x\}$, $\{G_b^y\}$

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(we want to rigorously deduce it!)

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or that the measurements are projective
(we want to rigorously deduce it!)

(ii) often only promised some Bell violation

$$\sum_{abxy} c_{ab}^{xy} \Pr[a,b|x,y] = \beta$$

Self-testing

Measurement: resolution of $\mathbb{1}$ into positive semidefinite operators

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This mapping is one-to-one: any A such that

$$A = A^\dagger \quad \text{and} \quad -\mathbb{1} \leq A \leq \mathbb{1}.$$

defines a valid measurement

Self-testing

Example: the CHSH inequality [1, 2]

$$\beta := \mathrm{tr}\left(W\rho_{AB}\right) \quad \text{for} \quad W := A_0 \otimes (B_0 + B_1) + A_1 \otimes (B_0 - B_1)$$

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$$|\Phi_{A'B'}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

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canonical realisation

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Inherent limitations

- cannot see auxiliary systems (ignored by measurements)

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"CHSH is rigid"

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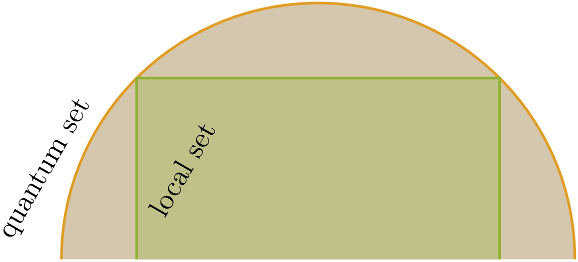
Rigidity is related to the **geometry of the quantum set of correlations**, e.g. for the CHSH inequality we have



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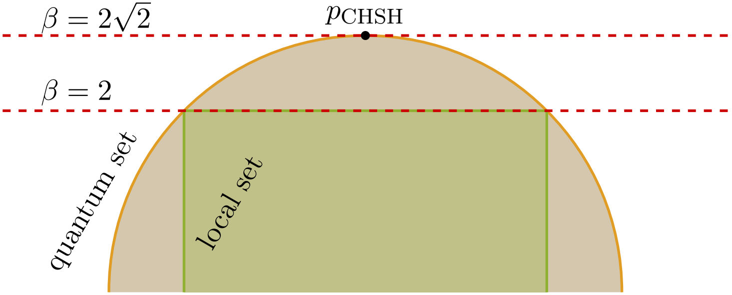

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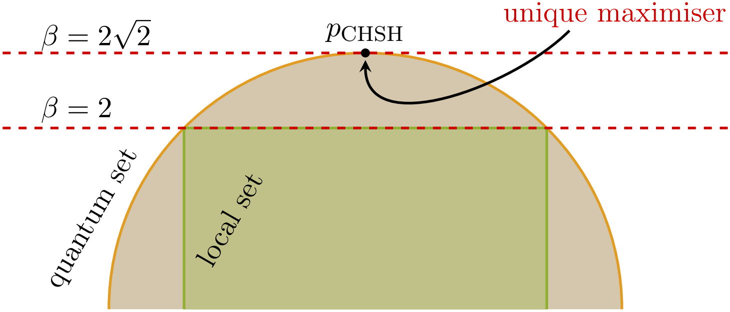

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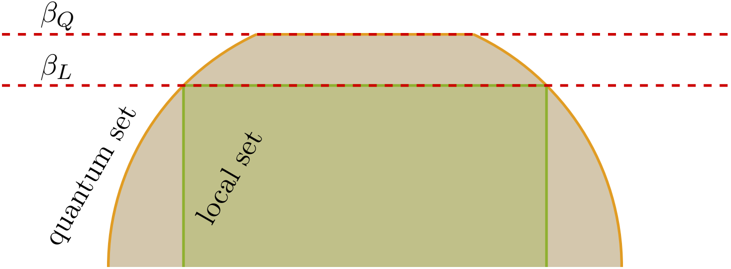

no unique maximiser $\implies$ no rigidity statement

Self-testing

Rigidity has been shown for:

- CHSH inequality [1, 2]
- tilted/biased CHSH inequality [6, 7, 8]
- chained Bell inequalities [9]
- tripartite Mermin inequality [10], MABK inequalities [8]
- magic square [11] and magic pentagram game [12]
- ...

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Exciting, can I see it in an **experiment**?

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In such a case, we would still expect that the quantum realisation must somehow **resemble** the canonical one

What is the right **mathematical formulation** of this statement?

Robust self-testing

Approach 1 (generic): require all equalities to hold up to some ε

$$\begin{aligned}
\beta = 2\sqrt{2} \quad &\Longrightarrow \quad \rho_{AB} = U(\Phi_{A'B'} \otimes \tau_{A''B''})U^\dagger, \\
\beta = 2\sqrt{2} - \varepsilon \quad &\Longrightarrow \quad \left\| \rho_{AB} - U(\Phi_{A'B'} \otimes \tau_{A''B''})U^\dagger \right\|_1 \leq f(\varepsilon).
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In such a stringent formulation $f(\varepsilon)$ grows very fast, non-trivial statement **only for almost maximal violations** (for CHSH only if $\varepsilon < 10^{-4}$)

Might be good enough for complexity-theoretic applications, but is it relevant from the physics point of view?

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Approach 2 (specific): choose one particular property and use a measure tailored to certify that property

Robust self-testing

Robust certification of quantum states

$$\rho_{AB} = U(\Psi_{A'B'} \otimes \tau_{A''B''})U^\dagger$$

$$\Longleftrightarrow$$

$$\exists \Lambda_A : A \to A',$$

$$\Lambda_B : B \to B'$$

$$\text{s.t.} \quad (\Lambda_A \otimes \Lambda_B)(\rho_{AB}) = \Psi_{A'B'}$$

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If cannot extract a **perfect** copy, then...?

Robust self-testing

Extractability of $\Psi_{A'B'}$ from ρ_{AB} [13, 14]

$$\Xi(\rho_{AB} \to \Psi_{A'B'}) := \max_{\Lambda_A, \Lambda_B} F\big((\Lambda_A \otimes \Lambda_B)(\rho_{AB}), \Psi_{A'B'}\big)$$

local extraction channels

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Obs1: $\Xi(\rho_{AB} \to \Psi_{A'B'}) = 1 \iff \rho_{AB} = U(\Psi_{A'B'} \otimes \sigma_{A''B''})U^\dagger$

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Obs2: $\Xi(\rho_{AB} \to \Psi_{A'B'}) \in [\lambda_{\max}^2, 1]$

largest Schmidt coefficient

Robust self-testing

Extractability is an operational quantity – once we have a good quality singlet, we can use it for any task for which a perfect singlet can be used

Given a state $\sigma_{AB} \in \mathcal{S}(\mathbb{C}^d \otimes \mathbb{C}^d)$ the **singlet fraction** is defined as

$$\max_{U=U_A \otimes U_B} F\big(U\sigma_{AB}U^\dagger, \Phi^d_{AB}\big)$$

for $\Phi^d = \frac{1}{\sqrt{d}} \sum_j |j\rangle|j\rangle$

Singlet fraction captures how useful σ_{AB} is for **teleporting a qudit** [15]

Extractability is a slightly more general notion (can deal with dimension mismatch), but has similar operational significance

Robust self-testing

In this formulation the goal is to derive **lower bounds**

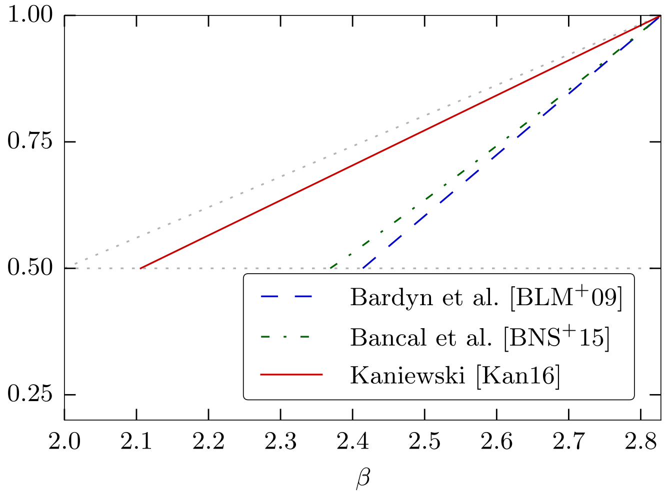
$$\Xi(\rho_{AB} \to \Psi_{A'B'}) \geq f(\beta)$$

The bound is **nontrivial** if

$$f(\beta) > \lambda_{\max}^2$$

Robust self-testing

Example 1: CHSH inequality ($\beta_C = 2$ and $\beta_Q = 2\sqrt{2}$) [13, 16, 14]
Lower bounds on extractability of $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$



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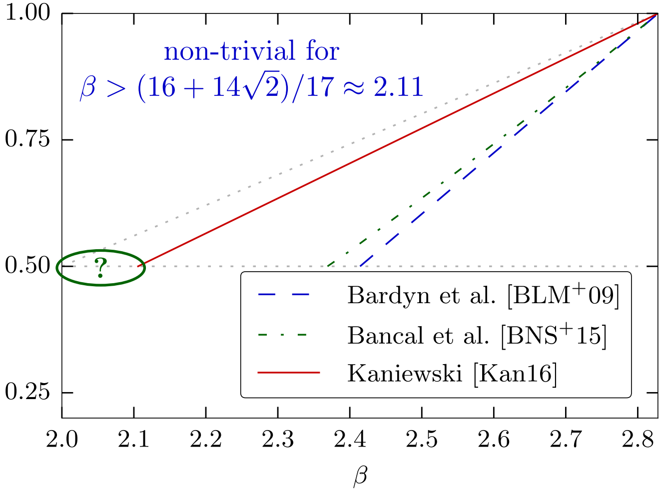

Robust self-testing

Example 2: Mermin inequality ($\beta_C = 2$ and $\beta_Q = 4$) [17, 14]
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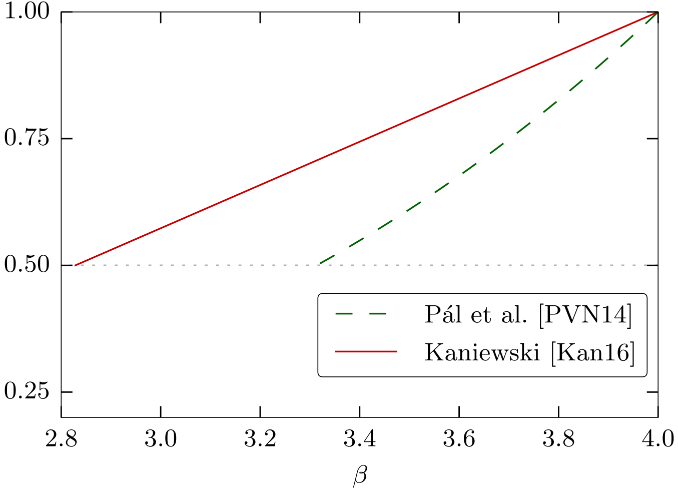

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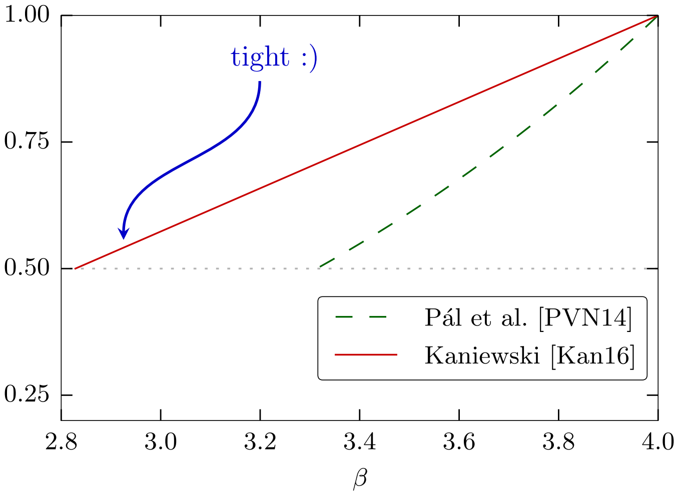

Robust self-testing

Certification of measurements

The optimal CHSH measurements

$$A_0 = U_A \big(\sigma_x \otimes \mathbb{1}\big) U_A^\dagger$$
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Equivalent formulation in terms of algebraic relations

$$A_0^2 = A_1^2 = \mathbb{1} \qquad \text{(projectiveness)}$$
$$A_0 A_1 + A_1 A_0 = 0 \qquad \text{(anticommutation)}$$

Robust self-testing

Idea: instead of certifying closeness to the canonical realisation certify **algebraic relations between observables** [8]

Convenient because

- it is clear how to measure "approximate" satisfaction of algebraic relations
- such quantities appear naturally in the analysis of the Bell operator
- non-trivial statements can be made for arbitrarily small violations
- can be used to guarantee uncertainty (useful for DI cryptography)

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This might not be the **ultimate answer**, but for binary observables these quantities have all the **desired properties**

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Summary and open questions

Summary:

- self-testing = certification of bi- or multipartite quantum systems under minimal assumptions
- direct link between the macroscopic and microscopic worlds
- insight into the geometry of the quantum set of correlations
- applications for device-independent cryptography

Summary and open questions

Open questions:

- which Bell inequalities are rigid and why? how generic is this phenomenon?
- all bipartite pure states can be self-tested [18], but some tripartite cannot: why?
- which arrangements of measurements can be self-tested and how?
- what is the "correct" formulation for robust self-testing of measurements?


So you can really certify quantum systems without trusting the devices at all?
Yes, Pooh, quantum mechanics is very strange and nobody really understands it, but let's talk about it another day. . .
THE END

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