Self-testing of qutrit systems

Jędrzej Kaniewski

QMATH, Department of Mathematical Sciences

University of Copenhagen, Denmark

joint work with

Antonio Acín, Remigiusz Augusiak, Flavio Baccari, Alexia Salavrakos, Ivan Šupić, Jordi Tura

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Outline

- Bell nonlocality
- Self-testing
- Sum-of-squares decomposition
- Example: CHSH inequality
- Result 1: SATWAP inequality
- Result 2: generalised CHSH inequality

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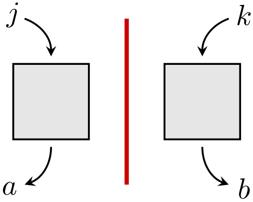
Bell nonlocality

Bell scenario



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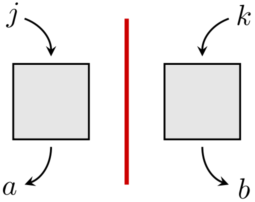

$P(a, b|j, k)$

Assume that $P \in \mathcal{Q}$ is **quantum**

$$P(a, b|j, k) = \text{tr}\left[(F_a^j \otimes G_b^k)\rho_{AB}\right].$$

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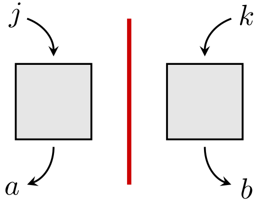

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$$P(a,b|j,k) = \mathrm{tr}\left[(F_a^j \otimes G_b^k)\rho_{AB}\right].$$

Def.: $P \in \mathcal{L}$ is **local** if

$$P(a,b|j,k) = \sum_\lambda p(\lambda)\, p_A(a|j,\lambda)\, p_B(b|k,\lambda).$$

Bell: $\mathcal{L} \subsetneq \mathcal{Q} \iff$ " quantum mechanics is (Bell) **nonlocal** "

Bell nonlocality

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$$\langle C, P \rangle := \sum_{abjk} c_{abjk} P(a, b|j, k)$$

and

$$\beta_{\mathcal{L}} := \max_{P \in \mathcal{L}} \langle C, P \rangle \quad \text{(local value)}$$

$$\beta_{\mathcal{Q}} := \max_{P \in \mathcal{Q}} \langle C, P \rangle \quad \text{(quantum value)}$$

(suppose $\beta_{\mathcal{L}} < \beta_{\mathcal{Q}}$)

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Bell violation: $\langle C, P \rangle > \beta_{\mathcal{L}} \implies P \notin \mathcal{L}$

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Obs.: Separable states give local statistics (for all measurements)

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Nonlocality $\implies$ **entanglement**

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can we make this connection more explicit/rigorous?

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Given $P(a,b|j,k) = \mathrm{tr}\left[(F_a^j \otimes G_b^k)\rho_{AB}\right]$

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$$\langle C, P \rangle = \beta$$

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might seem like a hopeless task...

...but often can deduce **essentially everything**!

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3. Then $\beta_Q \leq c$ because for all quantum realisations

$$\langle C, P \rangle = \mathrm{tr}(W \rho_{AB}) \leq c\, \mathrm{tr}(\rho_{AB}) = c$$

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A: Write difference as sum of squares

$$c\,\mathbb{1} - W \geq \sum_j L_j^\dagger L_j .$$

(operators L_j depend on measurement operators)

- if $\beta_Q = c \implies$ sum-of-squares (SOS) decomposition is **tight**

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$$W := A_0 \otimes (B_0 + B_1) + A_1 \otimes (B_0 - B_1)$$

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$$L_0 = A_0 \otimes \mathbb{1} - \mathbb{1} \otimes \frac{B_0 + B_1}{\sqrt{2}}$$

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$$W = \frac{1}{\sqrt{2}}\left[(A_0^2 + A_1^2) \otimes \mathbb{1} + \mathbb{1} \otimes (B_0^2 + B_1^2) - (L_0^\dagger L_0 + L_1^\dagger L_1)\right]$$

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- $W \leq 2\sqrt{2}\,\mathbb{1}$ and $\beta_{\mathcal{Q}} = 2\sqrt{2}$, so the SOS decomposition is **tight**

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if ρ_A and ρ_B are full-rank, then

$$A_0^2 = \mathbb{1} \implies \left(\frac{B_0 + B_1}{\sqrt{2}}\right)^2 = \mathbb{1} \implies \{B_0, B_1\} = 0$$

Example: the CHSH inequality

- the relation determines form of observables

$$B_0^2 = B_1^2 = \mathbb{1} \quad \text{and} \quad \{B_0, B_1\} = 0 \implies \begin{array}{l} B_0 = U_B(\sigma_x \otimes \mathbb{1})U_B^\dagger \\ B_1 = U_B(\sigma_z \otimes \mathbb{1})U_B^\dagger \end{array}$$

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Self-testing (rigidity) statement for CHSH: if $\beta = 2\sqrt{2}$ then

$$\begin{array}{ll} A_0 = U_A(\sigma_x \otimes \mathbb{1})U_A^\dagger & B_0 = U_B(\sigma_x \otimes \mathbb{1})U_B^\dagger \\ A_1 = U_A(\sigma_z \otimes \mathbb{1})U_A^\dagger & B_1 = U_B(\sigma_z \otimes \mathbb{1})U_B^\dagger \end{array}$$

and

$$\rho_{AB} = U(\Phi_{A'B'} \otimes \tau_{A''B''})U^\dagger \quad \text{for} \quad U := U_A \otimes U_B$$

Example: the CHSH inequality

Strategy:

1. Find tight SOS decomposition
2. Deduce algebraic relations between local observables
3. Deduce their exact form (up to unitaries and extra degrees of freedom)
4. Construct Bell operator and find eigenspace corresponding to $\beta_{\mathcal{Q}}$

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Result 1: SATWAP inequality

- maximally violated by **maximally entangled state** and **CGLMP measurements** (Remik's talk)
- CGLMP measurement in dimension d for $\phi \in [0, 2\pi]$

$$|e_j^\phi\rangle := \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \omega^{(j-\phi)k} |k\rangle \quad \text{for} \quad \omega := \exp(2\pi i/d)$$

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- we look at **2** inputs and **3** outputs, optimal angles $\phi_0 = 0$, $\phi_1 = 1/2$; computing $|\langle e_{j'}^{1/2}|e_j^0\rangle|$ gives

$j \backslash j'$	0	1	2
0	2/3	2/3	1/3
1	1/3	2/3	2/3
2	2/3	1/3	2/3

not mutually unbiased

Result 1: SATWAP inequality

- SOS written in terms of observables (unitary for projective measurements)

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- construct Bell operator $\implies$ ...

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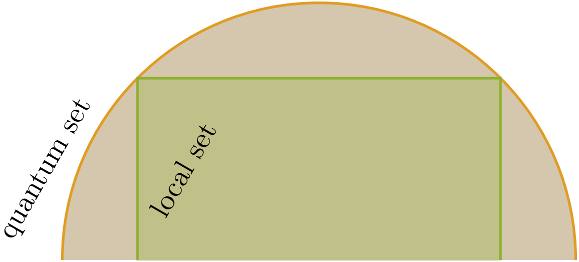
Cor. 1: SATWAP functional has a unique maximiser



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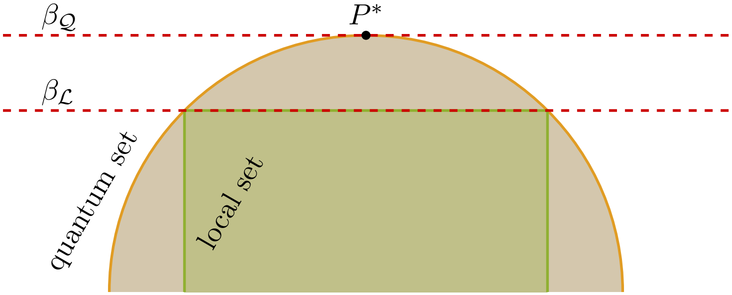

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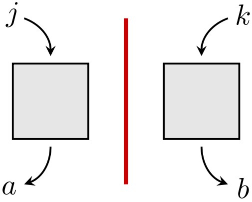

$$P(a, b|j, k)$$

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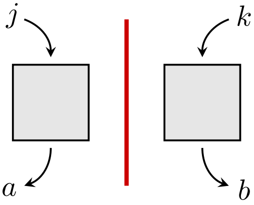

$P(a, b|j, k)$

CHSH: $a, b, j, k \in \{0, 1\}$

win $\iff a \oplus b \oplus jk = 0$

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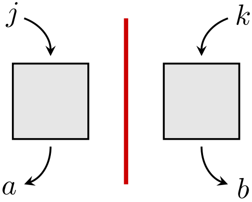

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CHSH$_d$: $a, b, j, k \in \{0, 1, \ldots, d-1\}$

win $\iff a + b + jk \equiv 0 \mod d$

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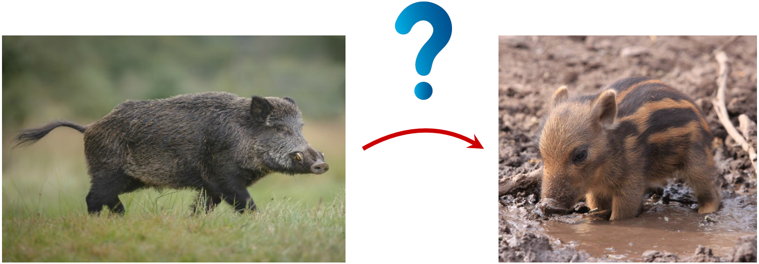

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  for $\lambda_{n,d} \in \mathbb{C}$, $|\lambda_{n,d}| = 1$

- for the right choice of $\lambda_{n,d}$ the quantum value can be computed **analytically** (tight SOS decomposition)!

Result 2: generalised CHSH inequality

- quantum realisation achieving the quantum value

$$|\Phi\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle|j\rangle$$

$$B_k = \omega^{k(k+1)} X Z^k$$

$$A_j = \dots$$

- the observables correspond to d distinct bases which are pairwise mutually unbiased
- for $d = 3$ SOS relations allow us to prove self-testing!

Result 2: generalised CHSH inequality

For $d = 3$ the phases are:

$$\lambda_{0,d} = 1, \quad \lambda_{1,d} = e^{-i\pi/18}, \quad \lambda_{2,d} = e^{+i\pi/18}$$
$$(e^{-i\pi/18} \approx 0.9849 - 0.1737i \approx 1)$$

- SOS + algebra $\implies$ **projectivity** and
$$B_0^\dagger = -\omega\{B_1, B_2\} \quad \text{(and perm.)}$$
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- this is sufficient to deduce the form of observables...
  ...except that now there are two inequivalent solutions

$$(B_0, B_1, B_2) \not\equiv (B_0^{\mathrm{T}}, B_1^{\mathrm{T}}, B_2^{\mathrm{T}}) \quad \textbf{not unitarily equivalent}$$
$$(\sigma_x, \sigma_y, \sigma_z) \not\equiv (\sigma_x, -\sigma_y, \sigma_z)$$

- local Hilbert spaces decompose into the "right-" and "left-handed" subspace and maximal violation possible only if Alice and Bob use opposite types!

Result 2: generalised CHSH inequality

Maximal violation certifies:

- $|\Phi\rangle = \frac{1}{\sqrt{3}} \sum_{j=0}^{2} |j\rangle|j\rangle$
- specific MUB measurements for each party
- the two measurements must be of the opposite type

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Corollaries:

- has unique maximiser in $\mathcal{Q}$
- certifies $\log 3$ bits of local randomness

Conclusions and open questions

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- extend SATWAP self-testing to arbitrary d
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- extend SATWAP self-testing to arbitrary d
- extend generalised CHSH self-testing to arbitrary prime d
- **robustness!**


So you can really certify quantum systems without trusting the devices at all?
Yes, Pooh, quantum mechanics is very strange and nobody really understands it, but let's talk about it another day...
THE END