Geometry of the quantum set of correlations

[arXiv:1710.05892]

Jędrzej Kaniewski

joint work with

Yu Cai, Koon Tong Goh, Yeong-Cherng Liang,

Valerio Scarani, Tamás Vértesi, Elie Wolfe, Xingyao Wu

QMATH, University of Copenhagen, Denmark

KCIK, Sopot

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QMATH Centre for the Mathematics of Quantum Theory

Outline

- What is the quantum set?
- What is self-testing?
- Unusual geometric features of the (bipartite) quantum set
- Tripartite scenarios
- Summary and open problems

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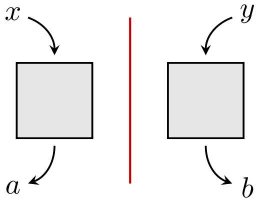

$$P(a, b|x, y)$$

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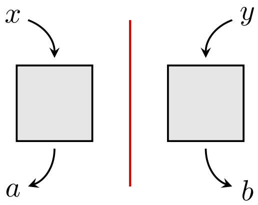

$P(a, b|x, y)$

Local-realistic theories give statistics of the form

$$P(a, b|x, y) = \sum_{\lambda} p(\lambda)\, p(a|x, \lambda)\, p(b|y, \lambda).$$

P belongs to the local set: $P \in \mathcal{L}$

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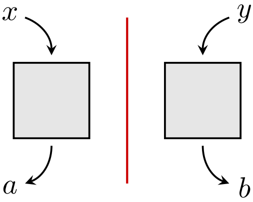

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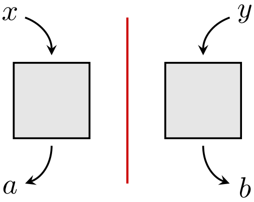

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[Bell'64]: quantum correlations can be **stronger!**

What is the quantum set?

The quantum set $\mathcal{Q}$: $P \in \mathcal{Q}$ if there exist:

- ρ_{AB}: bipartite state shared by the devices
- E_a^x: measurement operator of Alice for outcome a on input x
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such that

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How **big** can $\mathcal{Q}$ get?

What is the quantum set?

Observation: Quantum mechanics is **no-signalling**

$$\sum_b P(a,b|x,y) = \sum_b P(a,b|x,y') \quad \text{for all} \quad y, y'.$$

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[Popescu, Rohrlich'94]: correlations would be **even stronger**, define the no-signalling set $\mathcal{NS}$ as

$$P \in \mathcal{NS} \quad \text{if} \quad P(a,b|x,y) \geq 0 \quad \text{and obey no-signalling}$$

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Fact: $\mathcal{NS}$ is a **polytope** (finite number of linear inequalities)

What is the quantum set?

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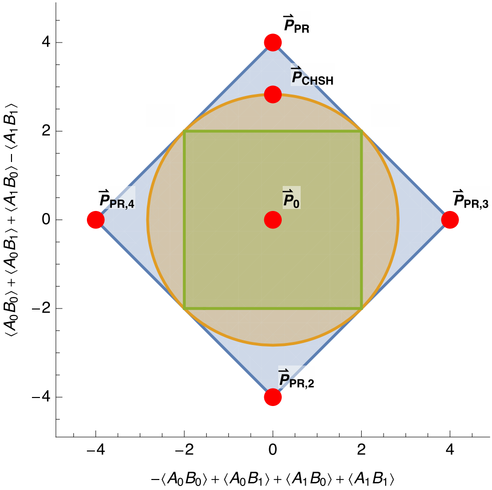

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"essence of nonlocality"

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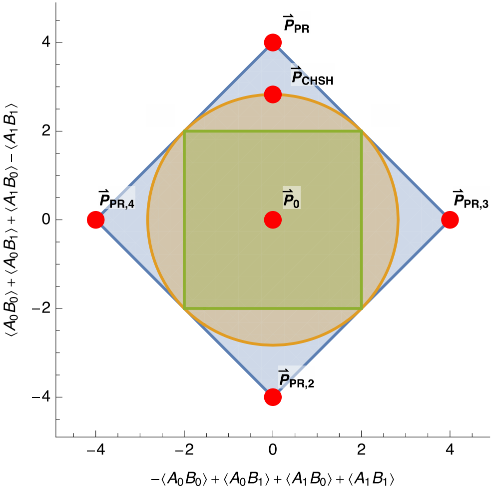

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often using
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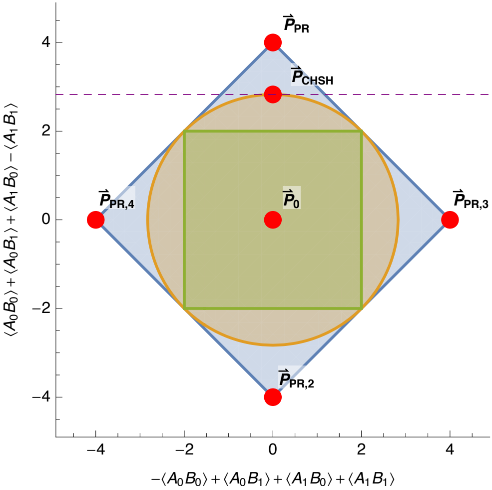

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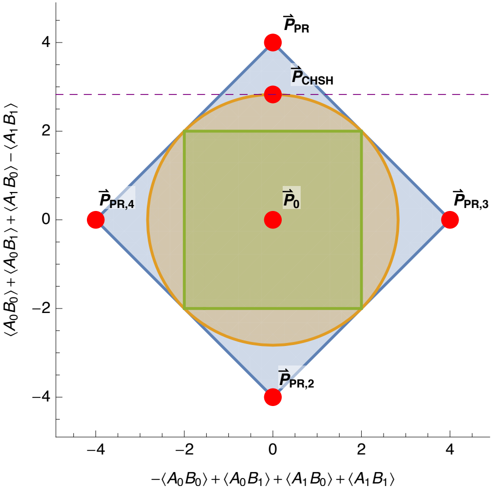
$B \cdot P := \sum_{abxy} c_{ab}^{xy} P(a, b|x, y)$

$\beta_{\mathcal{S}} := \max_{P \in \mathcal{S}} B \cdot P$



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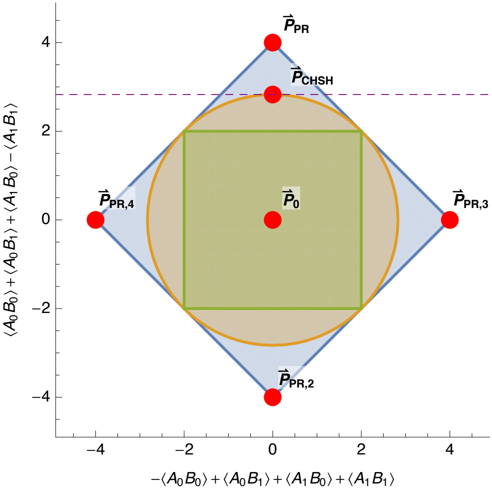

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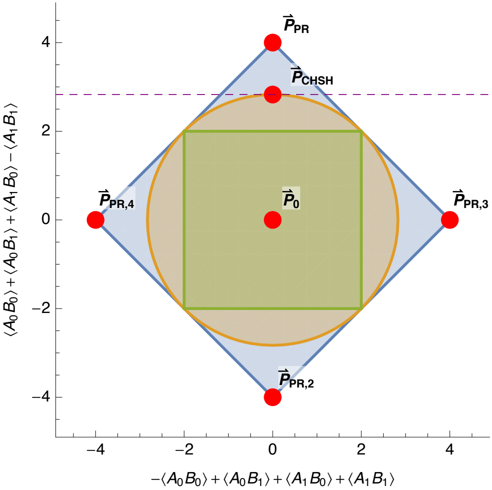

For CHSH function

$$\beta_{\mathcal{L}} = 2$$

$$\beta_{\mathcal{Q}} = 2\sqrt{2}$$

$$\beta_{\mathcal{NS}} = 4$$

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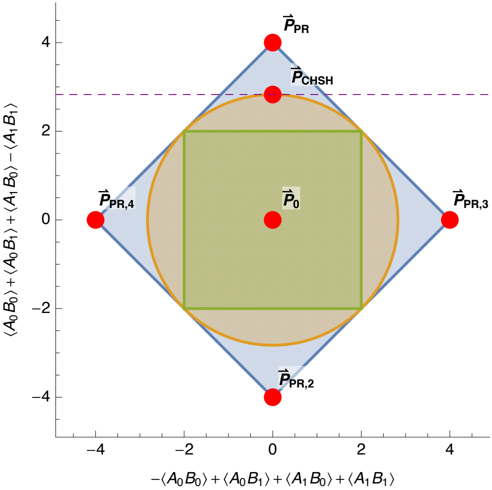

For CHSH function

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tells us **nothing**
about the geometry!

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What is self-testing?

Given some Bell violation

$$\beta = \sum_{abxy} c_{ab}^{xy} P(a,b|x,y)$$

arising from measuring a quantum system

$$P(a,b|x,y) = \mathrm{tr}\left[(E_a^x \otimes F_b^y)\rho_{AB}\right]$$

deduce properties of ρ_{AB}, (E_a^x), (F_b^y).

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Sounds **challenging**, but in some cases we can deduce **essentially everything**!

What is self-testing?

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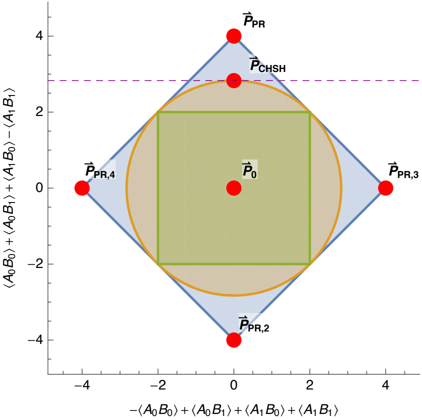
$$\rho_{AB} \simeq \Phi_{AB},$$

where Φ_{AB} = EPR pair and

$$A_0 \simeq \sigma_x, \quad A_1 \simeq \sigma_z,$$
$$B_0 \simeq \sigma_x, \quad B_1 \simeq \sigma_z.$$

“complete rigidity statement”

What is self-testing?



(1) maximal violation picks a single point $P \in \mathcal{Q}$

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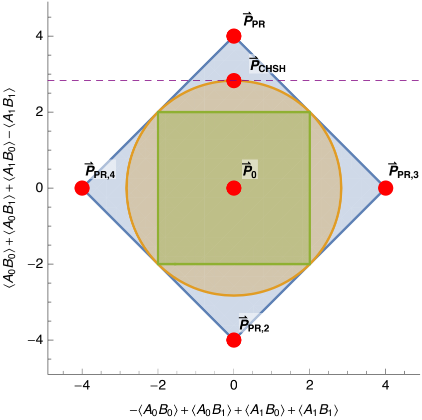

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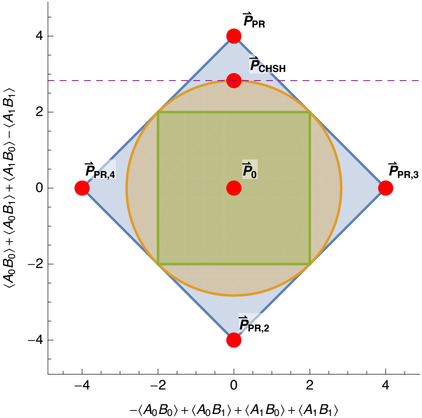

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Question: how generic is this situation?

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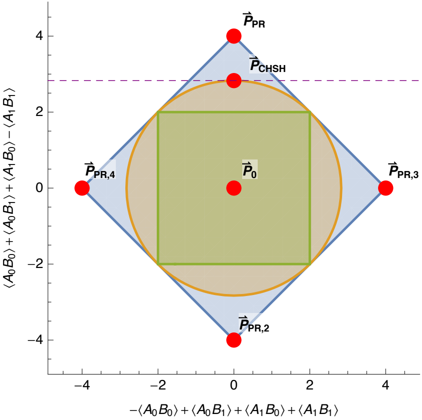

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Question: how generic is this situation?

Answer: already in the simplest Bell scenario (2 inputs, 2 outputs) things can get much more complicated...

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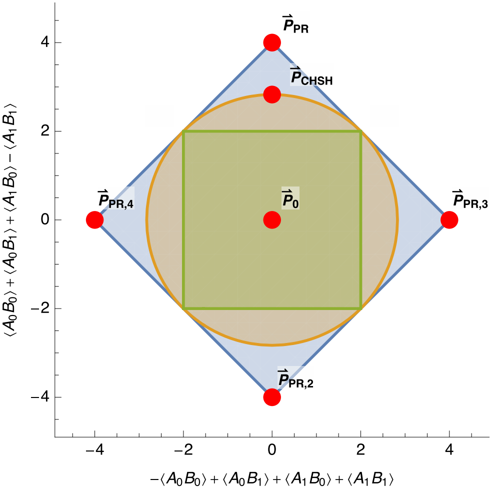

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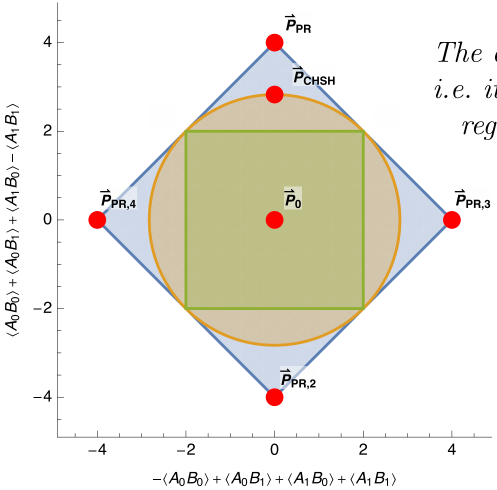
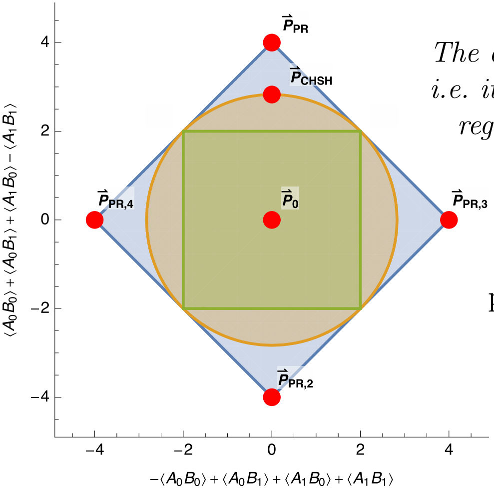


The quantum set is "round", i.e. it does not have any flat regions on the boundary

Unusual geometric features of the quantum set



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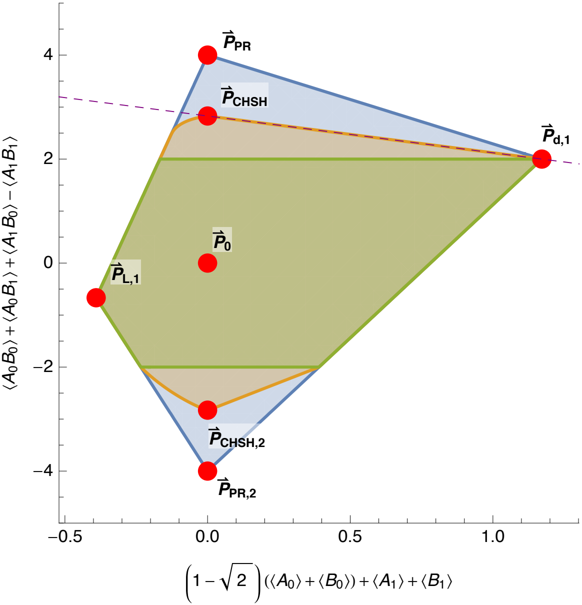
positivity facets. . .

Counterintuitive features of the quantum set



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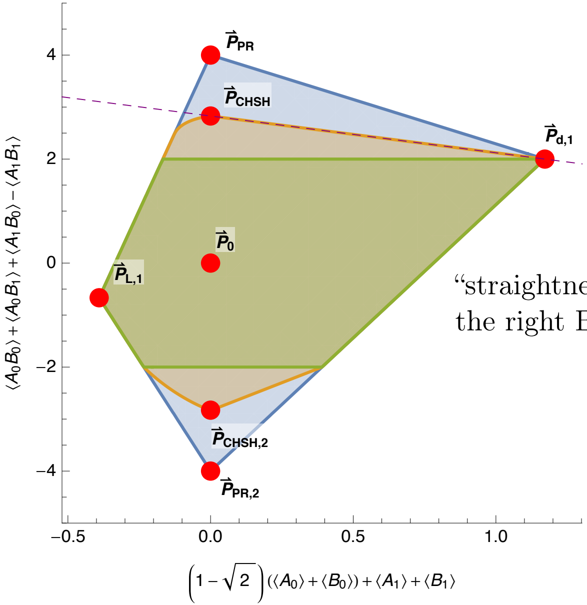

"straightness" certified by finding the right Bell function

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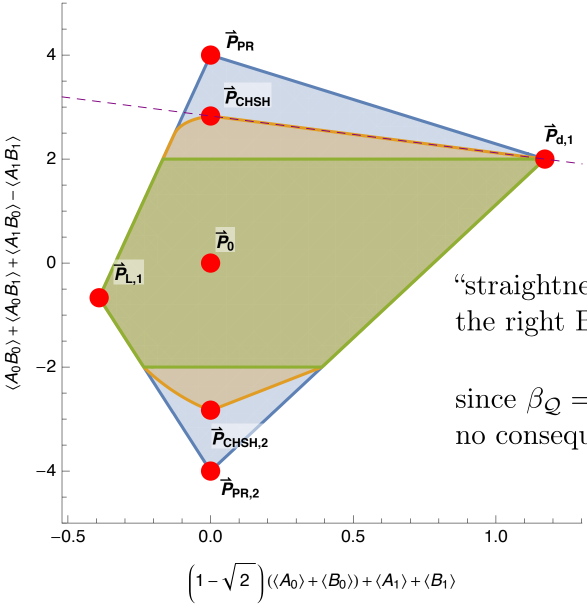

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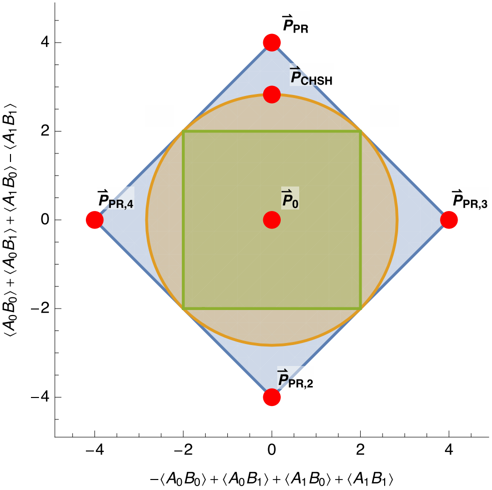
since $\beta_{\mathcal{Q}} = \beta_{\mathcal{L}} \implies$ no consequences for self-testing

Counterintuitive features of the quantum set



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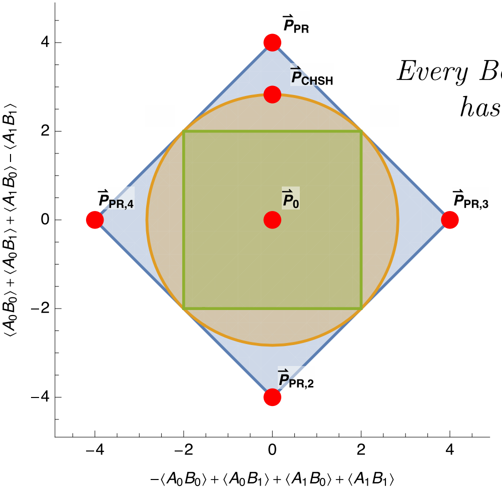
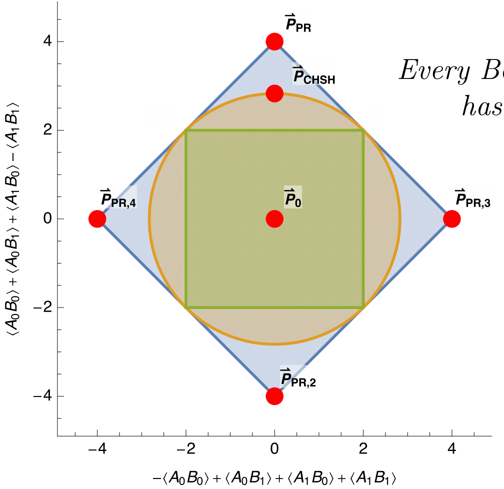


Every Bell function
has a unique maximiser

Counterintuitive features of the quantum set



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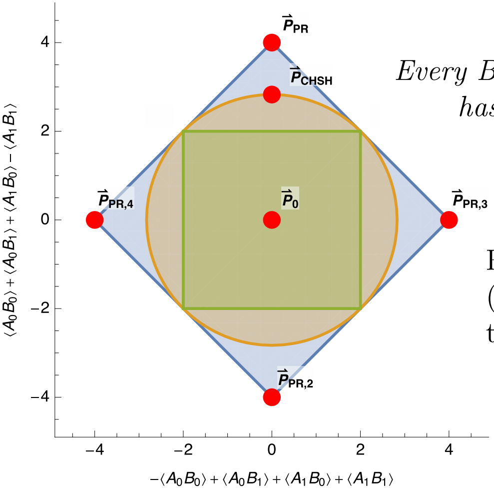

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Ramanathan, Mironowicz
(arXiv: 1704.03790)
tripartite scenario

Counterintuitive features of the quantum set

Example in the 3 input, 2 output scenario: take the I_{3322} function and remove the marginals

$\implies$ Bell function B s.t. $\beta_{\mathcal{L}} = 4$, $\beta_{\mathcal{Q}} = 5$, $\beta_{\mathcal{NS}} = 8$

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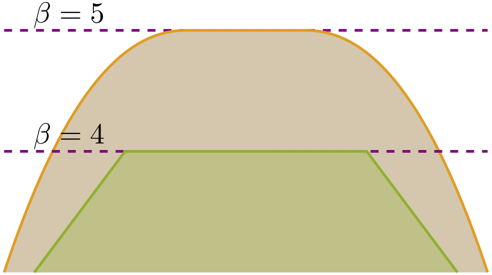

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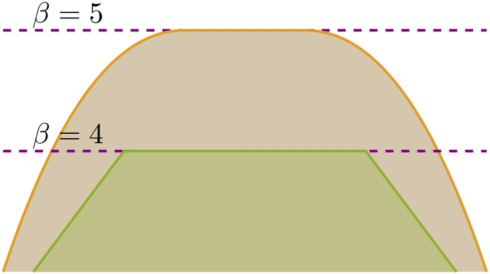
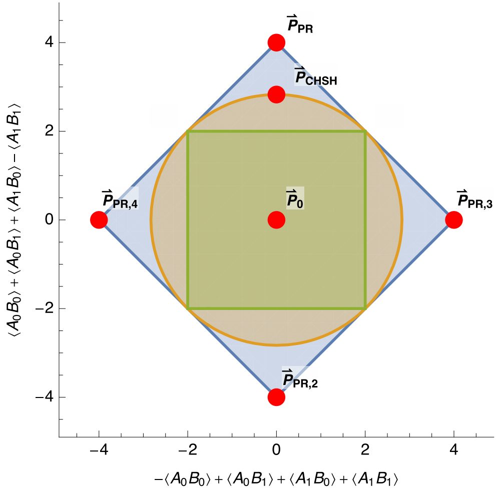


entire segment can be realised by projective measurements on $(|00\rangle + |11\rangle)/\sqrt{2}$

$\beta = 5$ **will not** certify observables, but **might** be sufficient to certify the state

Counterintuitive features of the quantum set



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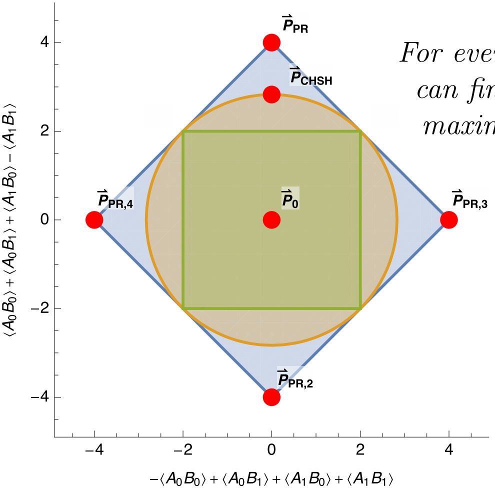
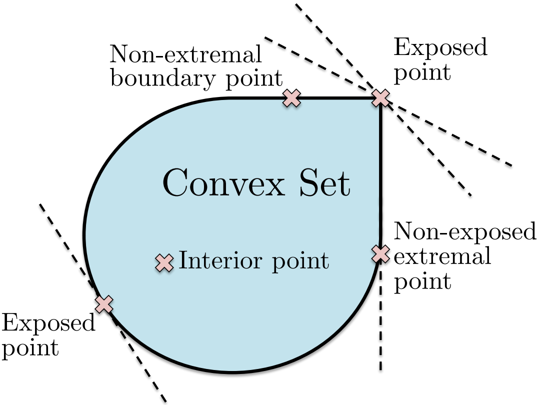


For every extremal point of $\mathcal{Q}$ we can find Bell function which is maximised only by that point.

Counterintuitive features of the quantum set

First page of any convex geometry textbook. . .



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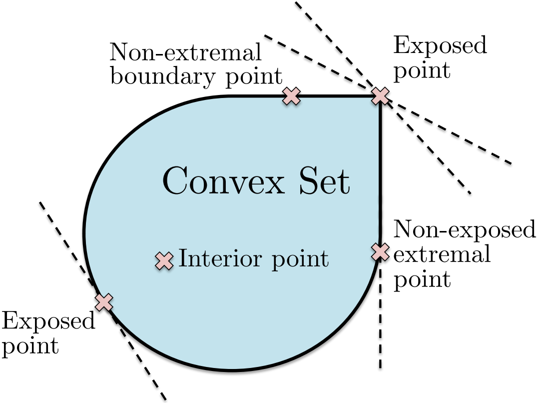
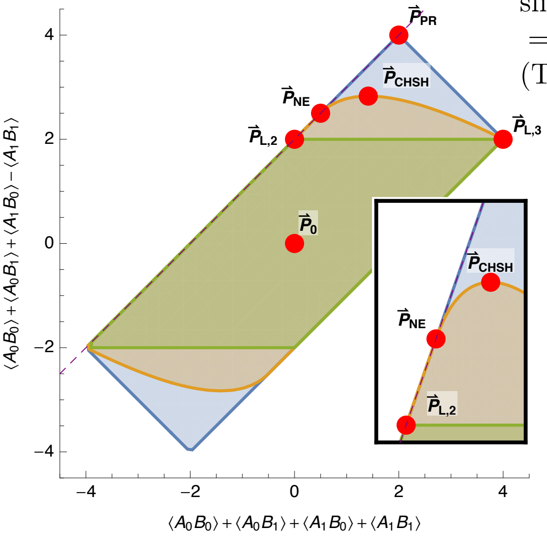


Fact: extremal but not-exposed points are **rare** (measure 0)

Counterintuitive features of the quantum set



slice of unbiased marginals
$\implies$ analytic characterisation
(Tsirelson-Landau-Masanes)

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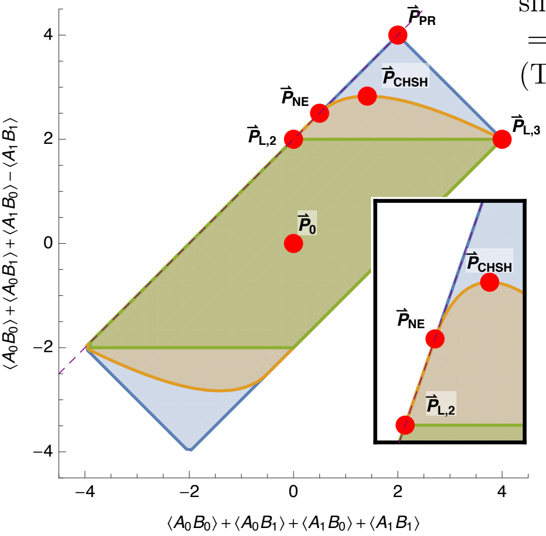

slice of unbiased marginals

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analytic proof of

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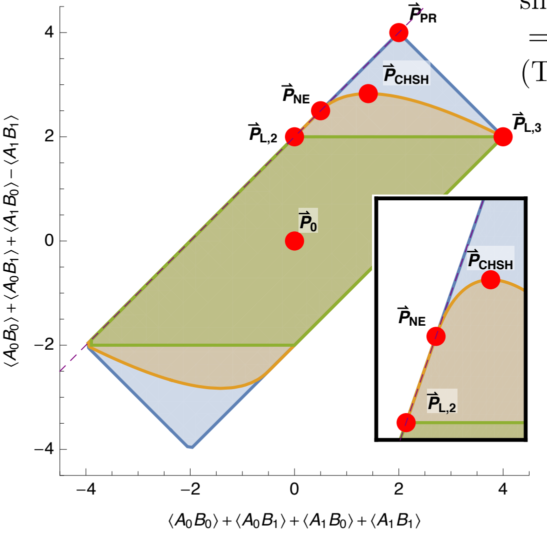

slice of unbiased marginals
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analytic proof of
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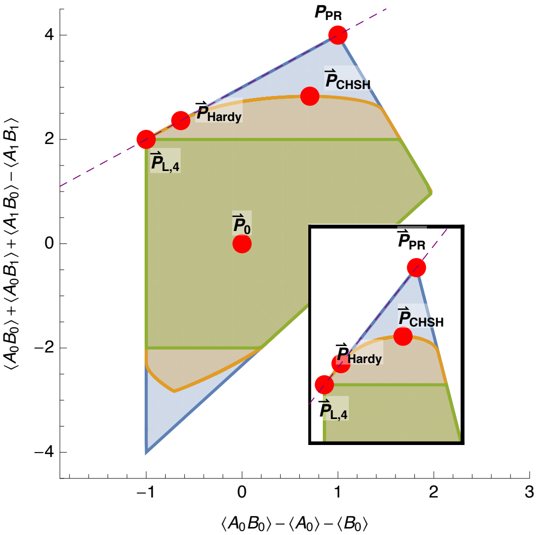
but known to be
extremal (self-test)

Counterintuitive features of the quantum set



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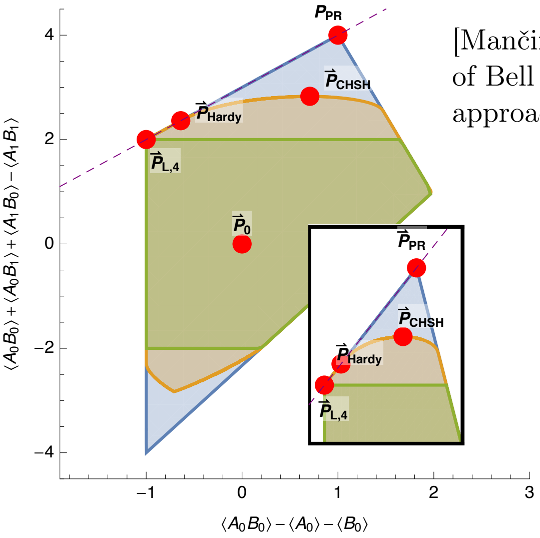

[Mančinska, Wehner'14]: sequence of Bell functions s.t. maximiser approaches P_{Hardy}

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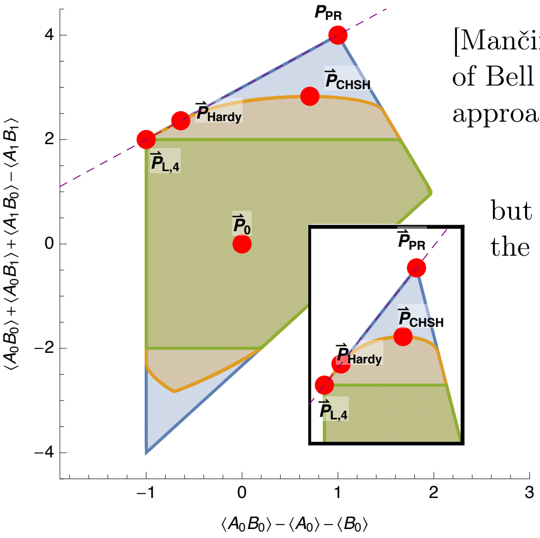

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but if we try to take the limit, the coefficients diverge

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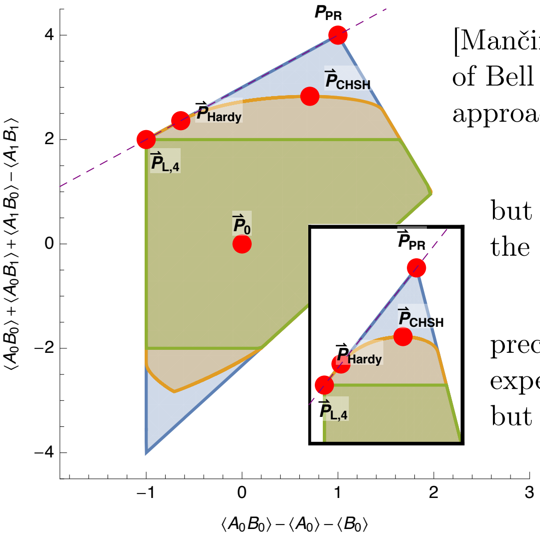

[Mančinska, Wehner'14]: sequence of Bell functions s.t. maximiser approaches P_{Hardy}

but if we try to take the limit, the coefficients diverge

precisely what one would expect from an extremal but not exposed point

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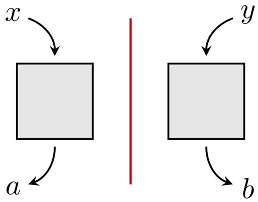
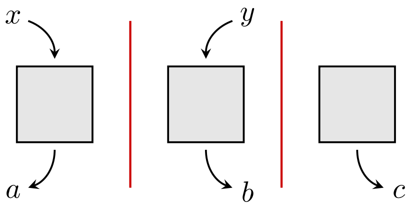
Tripartite scenarios



want to satisfy:
$a \oplus b = x \cdot y$

optimal violation certifies
$(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)/\sqrt{2}$

Tripartite scenarios



want to satisfy:
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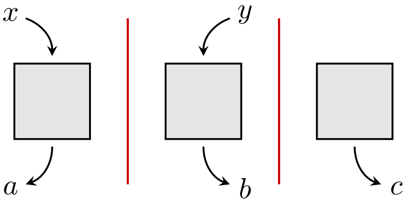
Tripartite scenarios



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What $P(a,b,c|x,y)$ achieve the quantum bound of $2\sqrt{2}$?

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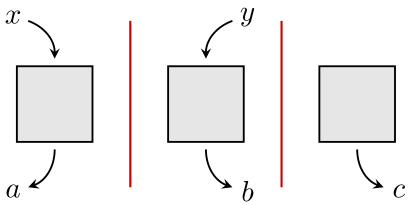

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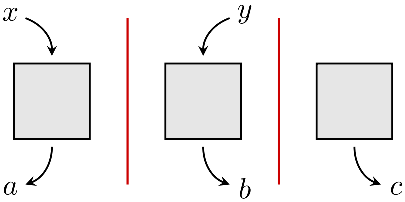

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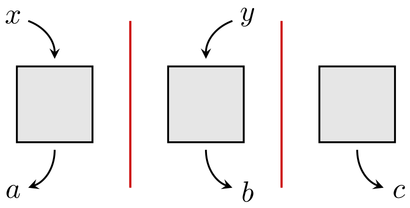

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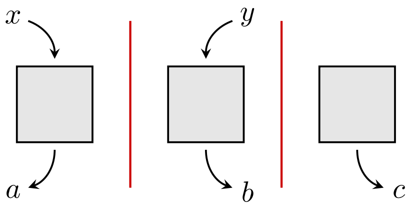
Proposition: if $P(a, b, c|x, y) \in \mathcal{Q}$ saturates the quantum bound, then P is a convex combination of (1) and (2)

$\implies$ the quantum face is a **line**!

Tripartite scenarios



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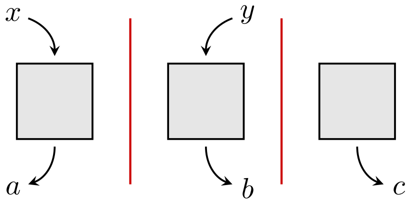

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$\implies$ we must have $(|0\rangle_A|0\rangle_{BC} + |1\rangle_A|1\rangle_{BC})/\sqrt{2}$, but not clear how it is split between Bob and Charlie…

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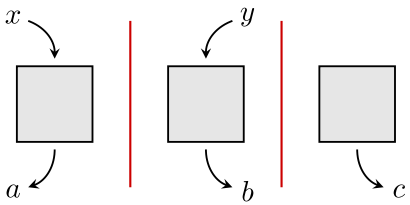

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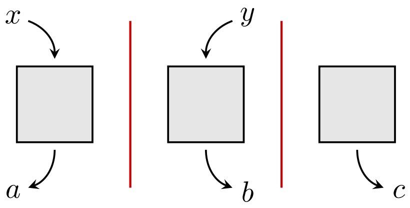

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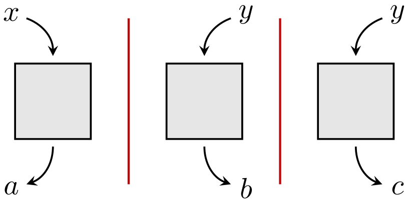
(iii) interior points can be achieved as convex combinations of bipartite entanglement, but also from a GHZ state $(|0\rangle_A|0\rangle_B|0\rangle_C + |1\rangle_A|1\rangle_B|1\rangle_C)/\sqrt{2}$

Tripartite scenarios



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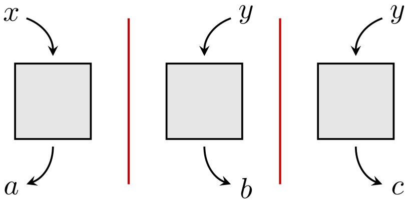

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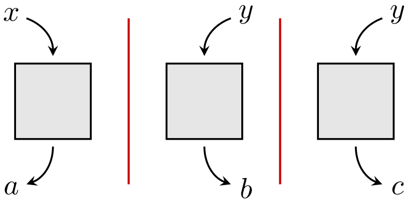

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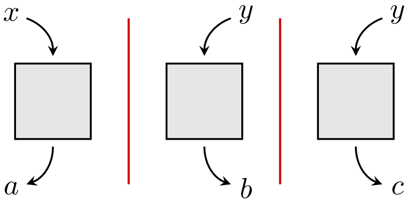

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Outline

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Open questions:

- Are there extremal points of $\mathcal{Q}$ which are not self-tests?
- What happens for a generic (chosen at random) Bell function/quantum face?


So the quantum set really has points which are extremal but not exposed?
Yes, Pooh, quantum mechanics is very strange and nobody really understands it, but let's talk about it another day…
THE END