Convexity of Q

Q_{fin}

P_0: $[P_a^x]_0$, $[Q_b^y]_0$, ρ_0 $\mathcal{H}_A \otimes \mathcal{H}_B$

P_1: $[P_a^x]_1$, $[Q_b^y]_1$, ρ_1 $\mathcal{H}_A \otimes \mathcal{H}_B$

$d_A := \max\{\dim \mathcal{H}_A, \dim \mathcal{H}_A\}$

$d_B := \max\{\dots\}$

both realisations can be embedded in $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$.

Consider realisation on:

$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$

Alice Bob

$\rho_{AB} = p\,|00\rangle\langle 00| \otimes \rho_0 + (1-p)\,|11\rangle\langle 11| \otimes \rho_1$

$P_a^x = |0\rangle\langle 0| \otimes [P_a^x]_0 + |1\rangle\langle 1| \otimes [P_a^x]_1$

$Q_b^y = \dots$

$P = p \cdot P_0 + (1-p) \cdot P_1$

ρ_{AB} is of higher dimension than ρ_0, ρ_1

necessary?

Q_{fin} is convex

$\Rightarrow Q = \mathrm{clo}(Q_{fin})$ is also convex

NS:

(1) positivity $P(ab|xy) \geq 0$

(2) norm $\sum_{ab} P(ab|xy) = 1$ $\forall x,y$

(3) $\sum_b P(ab|xy) = \sum_b P(ab|xy')$ $\forall y, y'$

$a = b \Leftrightarrow a \geq b \wedge b \geq a$

Compactness $\Leftrightarrow$ closed and bounded

$P \in \mathbb{R}^{n^2 k^2}$

(1) bounded $|P(ab|xy)| \leq 1$

$\Updownarrow$

$\|P\|_\infty \leq 1$

(2) closedness

$Q = \mathrm{clo}(Q_{fin})$ closed

NS - finite intersection of closed half-spaces

$\mathcal{L}$ - we will see soon

Polytope: compact convex set with a finite number of extremal points

vertices

$\mathcal{L}$: $r_A(a|x,\lambda) \in [0,1]$

wlog deterministic r_A ($r_A \in \{0,1\}$)

all randomness can be "pushed back" into λ

$P = \sum_\lambda p(\lambda) \cdot r_A(a|x,\lambda) \cdot r_B(b|y,\lambda)$

r_A: $[k] \to [n]$ n^k distinct functions

in total: $n^k \cdot n^k = n^{2k}$

$P = \sum_{j=1}^{n^{2k}} p_j \cdot D_j$

$P(ab|xy) = \begin{cases} 1 & \text{if } a = f_A(x),\ b = f_B(y) \\ 0 & \text{otherwise.} \end{cases}$

$\Rightarrow \mathcal{L}$ is a polytope

NS: finite intersection of closed half-spaces

Fact: if a fin. intersection of closed half spaces yields a bounded set, then this is a polytope.

Corollary: every polytope has 2 equivalent descriptions

(a) vertices (V)

(b) half-space (H)

Vertices: $(0,0)$, $(0,1)$, $(1,0)$ — min. descriptions — $\mathbb{R}^2$

$x \geq 0$

$y \geq 0$

$x + y \leq 1$

$x \leq 1$ valid but not necessary

min. description is unique

Compute H desc of $\mathcal{L}$:

(1) $P(ab|xy) \geq 0$

(2) no-signalling

(3) facet Bell inequalities

Compute V desc. of NS:

(1) deterministic points

(2) extremal NS boxes

	$\mathcal{L}$	NS
V	deterministic points	deterministic points; extremal NS boxes
H	pos. facets; no-sign. cond; facet Bell ineq.	positivity facets; no-sign. cond.

$\mathcal{L} \subsetneq NS$

$\mathcal{L}$, Q, NS - conv. compact

Membership problem: given P decide whether $P \in S$

(1) NS: check positivity; check NS conditions — easy

$P(a|x) := \sum_b P(ab|x, y=1)$

(2) $\mathcal{L}$ $N = n^{2k}$

$\{D_j\}_{j=1}^N$

$\{q_j\}_{j=1}^N$ $q_j \geq 0$, $\sum_j q_j = 1$

s.t. $\sum_j q_j D_j = P$

$A\binom{x}{y} \leq b$ for $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$x \geq 0$ ... $y \geq 0$, $x + y \leq 1$

Linear program (LP)

if $c = 0$ "feasibility problem"

max $c \cdot x$

over $x \in \mathbb{R}^d$ n rows, d columns

subject $A^T \cdot x \leq b$ ← n rows

matrix, vector

compact way of writing several inequalities in a single equation, see example

in principle possible (algorithm exists), but $N = n^{2k}$ exponential in k

(3) Q

can P be realised in $\mathbb{C}^d \otimes \mathbb{C}^d$?

could parametrise realisations and explore this space

in principle possible but exponentially hard

you should repeat for all $d \geq 2$

infeasible

Convex geometry

(1) dimension of convex set S

def "dimension of the smallest affine subspace that contains S"

A $m \times n$ real matrix

$Ax = b$ $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ ($b \neq 0$)

affine subspace (set of x's that satisfy this)

closed under affine combinations:

$\sum_j \alpha_j \cdot x_j$ s.t. $\sum_j \alpha_j = 1$

(cf. $Ax = 0$ linear subspace closed under $\sum_j \alpha_j x_j$)

Example: Polytope in $\mathbb{R}^3$ with vertices: $(\pm 1, \pm 1, 1)$.

dim = 2, because contained in affine subspace $z = 1$, whose dimension is 2