Convex geometry

(2) Faces

F : convex, nonempty subset of S (convex set)

F is face iff for every $z \in F$ the existence of $x, y \in S$ $px + (1-p)y = z$ $p \in (0,1)$ $\Rightarrow x, y \in F$

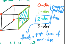
Faces have well-defined dimensions

0-dim faces – extremal points

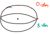
⋮

largest face – set S itself

Example: unit cube in $\mathbb{R}^3$



Example: unit ball in $\mathbb{R}^3$



(3) linear real functionals

$f: V \to \mathbb{R}$

$V = \mathbb{R}^n$, $f = \{f_j\}_{j=1}^n \in \mathbb{R}^n$

$$\langle f, x \rangle = \sum_{j=1}^n f_j x_j$$

Hyperplane separation thm (Hahn-Banach)

If S is a closed convex set in $\mathbb{R}^n$ and $x \notin S$. Then, there exists a linear functional and $c \in \mathbb{R}$ s.t.

$\langle f, y \rangle \le c$ $\quad y \in S$

$\langle f, x \rangle > c$

We will be interested in computing

$$\sup_{y \in S} \langle f, y \rangle$$

Corollary: Every closed and convex set can be described as an intersection of half-spaces.

Example: Describe unit disc in $\mathbb{R}^2$ in terms of half-spaces.

$x^2 + y^2 \le 1$ $\quad r = (x, y)$

$a_0 \cdot r \le 1$

$a_1 \cdot r \le 1$

$a \cdot r \le 1$ $\quad \forall$ unit vectors a

$a = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ $\quad \theta \in [0, 2\pi)$

$x \cdot \cos\theta + y \cdot \sin\theta \le 1 \quad \forall \theta$

V – H descriptions for polytopes

(1) Given V, want H

facet = intersection of hyperplane with S

every facet contains some vertices

For every subset of V: (perhaps of some minimal size)

(1) does it define a unique hyperplane?

(2) does P lie entirely on one side of this hyperplane?

• → facet • → no facet (cuts through P)

# of subsets of V is finite

⇒ # of facets of P is finite

⇒ # of half-spaces in H-desc is finite

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(2) Given H, want V

For subsets of H:

(1) Is the intersection a unique point?

(2) Is this point inside P?

Dimension of correlation sets

k measurement settings

n potential outcomes

$$D = 2(n-1) \cdot k + (n-1)^2 k^2$$

(1) Every $P \in NS$ can be parametrised by D real numbers. $\dim(NS) \le D$

(2) We give $(D+1)$ local points which are affinely independent $\dim(\mathcal{L}) \ge D$

$\mathcal{L} \subseteq Q \subseteq NS$

$\dim(\mathcal{L}) = \dim(Q) = \dim(NS) = D$

2 affinely ind. points ⇒ 1-dim. affine subspace

$[n] = \{1, \dots, n\}$

(1) $P(a|x)$ $\quad k(n-1)$ $\quad a \in [n-1], x \in [k]$

$P(b|y)$ $\quad k(n-1)$ $\quad b \in [n-1], y \in [k]$

$P(ab|xy)$ $\quad k^2(n-1)^2$ $\quad a, b \in [n-1], x, y \in [k]$

$\sum_{a=1}^n P(a|x) = 1 \Rightarrow P(n|x) = 1 - \sum_{a=1}^{n-1} P(a|x)$

Analogous for Bob $\Rightarrow P(n|y)$

E.g. $P(n, b|xy)$ $\quad b \le n-1$

Use NS condition

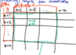
$$\sum_{a=1}^n P(ab|xy) = P(b|y)$$

$$P(n, b|xy) = P(b|y) - \sum_{a=1}^{n-1} P(ab|xy)$$

Analogous $P(a, n|x, y)$ $\quad a \le n-1$

Last step: $P(a|x) - \sum_{b=1}^{n-1} P(ab|xy)$

$P(n, n|xy)$ compute from normalisation



given for every pair (x, y)

(2) Specify $D+1$ local points

vectors in $\mathbb{R}^D$	I	II	III	IV
$P(a\|x)$ $a \le n-1$	0 0 ⋮ 0	0 0 1 0 0	0 0 ⋮ 0	0 0 ⋮ 1 ← $P(a=2\|x=2)$ 0
$P(b\|y)$ $b \le n-1$	0 0 ⋮ 0	0 0 ⋮ 0	0 0 ⋮ 1 0	0 0 ⋮ 1 ← $P(b=2\|y=3)$ 0
$P(ab\|xy)$ $a, b \le n-1$	0 0 ⋮ 0	0 0 ⋮ 0	0 0 ⋮ 0	0 0 1 ← $P(a=2, b=2\|x=2, y=3)$ 0
	1	$k \cdot (n-1)$	$k(n-1)$	$k^2(n-1)^2$

$D+1$

$\{x_0, x_1, \dots, x_n\}$ affinely independent

⇕

$\{\underbrace{x_1 - x_0}_{y_1}, \underbrace{x_2 - x_0}_{y_2}, \dots, \underbrace{x_n - x_0}_{y_n}\}$ linearly independent

$$\sum_j \alpha_j \cdot y_j = 0 \Rightarrow \alpha_j = 0$$