Advanced quantum information: entanglement and nonlocality

5. Homework sheet

Solutions should be submitted via email to Gabriel.Pereira@fuw.edu.pl by 23:59 on 22.06.2022. Please submit a single pdf file using “[AQI homework]” in the subject line.

Problem 1. Consider a family of probability points in the CHSH scenario which in the reduced coordinates are given by:

$$\begin{aligned}\langle A_x \rangle &= \langle B_y \rangle = 0, \\ \langle A_0 B_0 \rangle &= \langle A_1 B_0 \rangle = \cos\theta, \\ \langle A_0 B_1 \rangle &= \sin\theta, \\ \langle A_1 B_1 \rangle &= -\sin\theta\end{aligned}$$

for $\theta \in [0, \pi/2]$.

1. Check that these points satisfy the TLM criterion. For which values of θ are these points extremal? (Hint: recall that when the marginals are unbiased extremality is equivalent to saturating the TLM criterion and having at most 1 correlator of unit modulus)

2. Recall that the Γ matrix for the first level of the NPA hierarchy in the CHSH scenario is given by:

$$\Gamma = \begin{pmatrix} 1 & \langle A_0 \rangle & \langle A_1 \rangle & \langle B_0 \rangle & \langle B_1 \rangle \\ \langle A_0 \rangle & 1 & u & \langle A_0 B_0 \rangle & \langle A_0 B_1 \rangle \\ \langle A_1 \rangle & u^* & 1 & \langle A_1 B_0 \rangle & \langle A_1 B_1 \rangle \\ \langle B_0 \rangle & \langle A_0 B_0 \rangle & \langle A_1 B_0 \rangle & 1 & v \\ \langle B_1 \rangle & \langle A_0 B_1 \rangle & \langle A_1 B_1 \rangle & v^* & 1 \end{pmatrix},$$

where $u, v \in \mathbb{C}$ are the “unphysical” entries. Our goal is to find a pair of (u, v) which makes this matrix positive semidefinite for every value of θ.

   (a) Every submatrix of a positive semidefinite matrix is also positive semidefinite. By looking at the submatrix

   $$\begin{pmatrix} 1 & u \\ u^* & 1 \end{pmatrix}$$

   deduce the allowed range of u. Clearly, analogous argument applies to v.

   (b) By analysing what happens when a valid Γ matrix is replaced by $\frac{1}{2}(\Gamma + \Gamma^{\mathrm{T}})$ argue that without loss of generality u and v might be taken to be real.

   (c) Using a numerical package of your choice (Mathematica, Matlab, Octave, Python, etc) for each $\theta \in [0, \pi/2]$ find a pair (u, v) which makes the Γ matrix positive semidefinite. (Hint: for some values of θ the region we are looking for might be quite small; to find it you might want to look for (u, v) such that all the eigenvalues are larger than $-\varepsilon$ for, say, $\varepsilon = 0.1$ and see what happens as you decrease ε down to 0) Plot these values in a single plot, i.e. plot $u(\theta)$ and $v(\theta)$ as two separate curves. Can you guess analytic expressions for these curves?

Problem 2. Let $\{F_a\}_{a=1}^{n_A}$ and $\{G_b\}_{b=1}^{n_B}$ be two measurements on the same Hilbert space $\mathcal{H}$. We say that these measurements are compatible if there exists another measurement $\{H_{ab}\}$ where $a \in [n_A]$

and $b \in [n_B]$ such that

$$\sum_{b=1}^{n_B} H_{ab} = F_a \quad \text{for all} \quad a \in [n_A],$$
$$\sum_{a=1}^{n_A} H_{ab} = G_b \quad \text{for all} \quad b \in [n_B].$$

1. Argue that if $[F_a, G_b] = 0$, then $H_{ab} = F_a G_b$ constitutes a valid measurement. Why does this construction fail if the measurements do not commute?

2. Let $\mathsf{X}, \mathsf{Y}, \mathsf{Z}$ be the Pauli matrices and let $M := c_x \mathsf{X} + c_y \mathsf{Y} + c_z \mathsf{Z}$ for some $c_x, c_y, c_z \in \mathbb{R}$. By computing $\operatorname{tr} M$ and $\operatorname{tr} M^2$ compute the eigenvalues of M. Hence, determine for which values of $\alpha \in \mathbb{R}$ the operator $\alpha \mathbb{1} + M$ is positive semidefinite.

3. Consider the noisy version of the Pauli X and Z measurements defined by:

$$F_1 := \frac{1}{2}(\mathbb{1} - \eta \mathsf{X}), \quad F_2 := \frac{1}{2}(\mathbb{1} + \eta \mathsf{X}),$$
$$G_1 := \frac{1}{2}(\mathbb{1} - \eta \mathsf{Z}), \quad G_2 := \frac{1}{2}(\mathbb{1} + \eta \mathsf{Z}),$$

   where $\eta \in [0, 1]$ is the sharpness parameter. Determine for which values of $u, v \in \mathbb{R}$ the operators

$$H_{ab} = u\big[(-1)^a \mathsf{X} + (-1)^b \mathsf{Z}\big] + v\mathbb{1}$$

   constitute a valid parent measurement. Hence, compute the largest η for which $\{F_1, F_2\}$ and $\{G_1, G_2\}$ are compatible.