Advanced quantum information: entanglement and nonlocality

4. Homework sheet

Solutions should be submitted via email to Gabriel.Pereira@fuw.edu.pl by 15.06.2022.
Please submit a single pdf file using "[AQI homework]" in the subject line.

Problem 1. Consider a family of Bell functionals in the CHSH scenario given by

$$\beta = \alpha\langle A_0 B_0\rangle + \langle A_0 B_1\rangle + \alpha\langle A_1 B_0\rangle - \langle A_1 B_1\rangle, \tag{1}$$

where $\alpha \geq 0$ is a real parameter.

1. Find the local value of the Bell functional as a function of α.
2. Write down the Bell operator W_α corresponding to this Bell functional.
3. Write down W_α for the following observables of Alice:

$$A_0 = \cos a\,\mathsf{X} + \sin a\,\mathsf{Z}, \tag{2}$$
$$A_1 = \cos a\,\mathsf{X} - \sin a\,\mathsf{Z}, \tag{3}$$

   Given that $a \in [0, \pi/2]$ use the trivial bound $\|B_0\|, \|B_1\| \leq 1$ to write down an upper bound on $\|W_\alpha\|$ which only depends on a. Find the angle a^* which maximises this bound.

4. Given that correlation Bell inequalities are maximally violated by the maximally entangled state $|\Phi^+\rangle$ find the optimal choice of B_0 and B_1 (hint: it might be useful to write down the density matrix corresponding to $|\Phi^+\rangle$ in the Pauli basis). Verify that the upper bound derived above is achievable and hence must be equal to the quantum value of the Bell functional. For which values of α the local and quantum values differ?
5. Compute the statistics, i.e. all the marginals and correlators, for the optimal realisation derived in the previous steps.
6. The optimal measurements derived above are performed on the isotropic state:

$$\rho_{\text{iso}} := p\,\Phi^+ + (1-p)\frac{\mathbb{1}\otimes\mathbb{1}}{4}, \tag{4}$$

   where $p \in [0, 1]$ is a real parameter. Compute the observed value of the Bell functional as a function of p and α. For every $\alpha > 0$ derive the critical p^* at which the system no longer violates the local bound. Find the value of α which enables us to certify nonlocality of the largest class of isotropic states.

Problem 2. In Section 2.2 we discussed the projection of the quantum set onto the correlators. For our purposes an operation $\mathcal{P}$ is called a projection if it satisfies:

$$\mathcal{P}(x+y) = \mathcal{P}(x) + \mathcal{P}(y) \quad \forall x, y \quad \text{(linearity)}, \tag{5}$$
$$\mathcal{P}(\mathcal{P}(x)) = \mathcal{P}(x) \quad \forall x \quad \text{(idempotency)}. \tag{6}$$

1. Let $\mathcal{S}$ be a convex set and let $\mathcal{R} := \mathcal{P}(\mathcal{S})$. Show that $\mathcal{R}$ is convex.
2. Let $\mathcal{S} \subseteq \mathbb{R}^3$ be a convex set and let $\mathcal{R} \subseteq \mathbb{R}^2$ be its projection. By giving specific examples show that the following are possible:

   (a) an extremal point of $\mathcal{S}$ is projected onto a non-extremal point of $\mathcal{R}$,

(b) a non-extremal point of $\mathcal{S}$ is projected onto an extremal point of $\mathcal{R}$.

3. Prove that projecting a compact convex set is equivalent to projecting only its extremal points and then taking the convex hull (hint: use the Krein–Millman's theorem). Hence, conclude that the projection of a polytope is a polytope.

4. Find the projections of the local and no-signalling polytopes onto the correlators (the last 4 coordinates). Write down all their vertices and for every vertex determine how many vertices of the original polytope get projected to that vertex. Do you know what geometric solids in $\mathbb{R}^4$ these sets correspond to?

Problem 3. In this problem you are asked to compute and draw two specific 2-dimensional slices of the correlation sets. The simplest way to specify a slice is to express the entire probability point as an affine function of two real numbers which we denote by x and y. Then, the slice of a particular correlation set is represented by a subset of $\mathbb{R}^2$.

For polytopes the simplest approach is to take the linear constraints appearing in the minimal half-space descriptions and transform them into constraints on (x, y). For the quantum set you should use the TLM criterion because both slices correspond to unbiased marginals.

For each slice write down an analytic condition in terms of x and y for every correlation set and sketch all three correlation sets in a single drawing.

1. The first slice is given by:

$$\langle A_x \rangle = \langle B_y \rangle = 0, \tag{7}$$
$$\langle A_0 B_0 \rangle = x, \tag{8}$$
$$\langle A_0 B_1 \rangle = y, \tag{9}$$
$$\langle A_1 B_0 \rangle = y, \tag{10}$$
$$\langle A_1 B_1 \rangle = -x. \tag{11}$$

2. The second slice is given by:

$$\langle A_x \rangle = \langle B_y \rangle = 0, \tag{12}$$
$$\langle A_0 B_0 \rangle = x, \tag{13}$$
$$\langle A_0 B_1 \rangle = x, \tag{14}$$
$$\langle A_1 B_0 \rangle = x, \tag{15}$$
$$\langle A_1 B_1 \rangle = y. \tag{16}$$